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Permanent rotations of a body with a viscous filler suspended on a rod☆

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ABSTRACT

Problems of the existence, stability, and branching of the permanent rotations of a heavy, dynamically symmetrical rigid body suspended on a rod and which has an axisymmetric ellipsoidal cavity filled with a fluid are discussed. The phenomenological model of the friction of the fluid against the cavity wall proposed by Samsonov is used. All the trivial permanent rotations of the system and the non-trivial rotations that branch off from the trivial ones are found. Their stability and branching are investigated using a modified Routh's theory. The results obtained are presented in the form of an atlas of bifurcation diagrams.

Basic results on the dynamics and stability of the motion of bodies with cavities containing a fluid were obtained in Rumyantsev's work (see Ref. 1). Fundamental results on the dynamics of a body suspended on a rod have been obtained by several researchers (see, for example, Refs. 2–6).

1. Statement of the problem

Consider the problem of the motion of a dynamically symmetrical body that has a cavity completely filled with a viscous fluid and is suspended on a weightless non-deformable rod O_1O at the fixed point O_1. The elements are attached at the fixed point O_1 and the point of suspension O of the body by means of ideal ball joints. It is assumed that the point O lies on the dynamical axis of symmetry of the body (Fig. 1).

The cavity located within the body or shell has the form of an ellipsoid of revolution with semiaxes b_1, $b_2 = b_1$, b_3. It is assumed that the axis of symmetry of the cavity is the same as the dynamical axis of symmetry of the shell. The cavity is completely filled with a viscous fluid.

Let C be the centre of mass of the body–fluid system. We will introduce the moving system of coordinates $Cx_1x_2x_3$, whose axes are rigidly connected to the shell and are directed along its central principal axes of inertia, which are specified by the unit vectors $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$, where $\mathbf{i}_3 = \overrightarrow{CO}/|\overrightarrow{CO}|$ represents the dynamical axis of symmetry of the shell. Let m be the mass of the entire system, and let $\boldsymbol{\upsilon}$ be the absolute velocity vector of the point C. The system is situated in a uniform gravitational field $-mg\boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ is an ascending vertical unit vector. We will introduce the following notation: $-N\mathbf{e}$ is the tension force of the rod, where $\mathbf{e} = \overrightarrow{O_1O}/|\overrightarrow{O_1O}|$ is a unit vector directed along the rod, N is the projection of the tension force onto the vector $-\mathbf{e}$; $l = |\overrightarrow{O_1O}|$ is the length of the rod, $a = |\overrightarrow{CO}|$, and $\mathbf{a} = \overrightarrow{CO} = a\mathbf{i}_3$ is the radius vector of the point of attachment of the rod to the body relative to the centre of mass C.

We will use the method previously described in Ref. 7 to describe the motion of the system. The angular momentum vector $\mathbf{G}_s$ of the shell about to the centre of mass C is related to the angular velocity $\boldsymbol{\omega}$ of the body by the expression

$$\mathbf{G}_s = \mathbf{J}_s\boldsymbol{\omega}, \quad \mathbf{J}_s = \mathrm{diag}(A_s, A_s, C_s)$$

where $\mathbf{J}_s$ is the inertia tensor of the shell relative to the $Cx_1x_2x_3$ axis system, and A_s and C_s are the equatorial and axial moments of inertia of the shell about to the centre of mass of the system. We will assume that the state of the viscous fluid can be described by three components of the mean "vorticity" tensor $\boldsymbol{\Omega}$, which is an integral characteristic of the fluid in the cavity. Then the angular momentum vector of the

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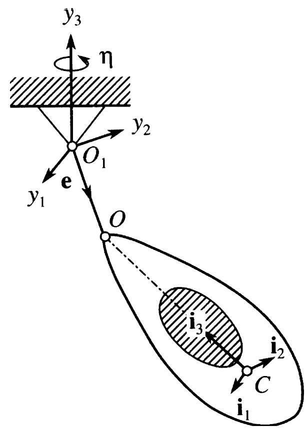

Fig. 1.

filler about to the centre of mass C can be represented in the form[1]

$$\mathbf{G}_l = \mathbf{J}^*\boldsymbol{\omega} + \mathbf{J}'\boldsymbol{\Omega}, \quad \mathbf{J}^* = \operatorname{diag}\{A^*, A^*, C^*\}, \quad \mathbf{J}' = \operatorname{diag}\{A', A', C'\}$$

where $\mathbf{J}^*$ is the inertia tensor of the so-called equivalent body (in the $Cx_1x_2x_3$ axes system), $\mathbf{J}'$ is the difference between the inertia tensor of the fluid $\mathbf{I} = \operatorname{diag}\{A_l, A_l, C_l\}$ and the inertia tensor of the equivalent body $\mathbf{J}^*$ relative to the $Cx_1x_2x_3$ axis system. The components of $\mathbf{I}$ and $\mathbf{J}^*$ depend on the mass of the fluid and the geometrical dimensions of the cavity and have been obtained in the form of exact formulae for some types of cavities, including ellipsoidal cavities (see, for example, Ref. 1). If we use m_s to denote the mass of the shell and m_l to denote the mass of the fluid, we have

$$A_s = m_s(\rho_1^2 + s^2), \quad C_s = m_s\rho_3^2, \quad A^* = \frac{m_l}{5}b_3^2\frac{(1-\delta^2)^2}{1+\delta^2}, \quad C^* = 0$$

$$A_l = \frac{m_l}{5}b_3^2(1+\delta^2), \quad C_l = \frac{2m_l}{5}b_3^2\delta^2$$

Here ρ_1, $\rho_2 = \rho_1$ and ρ_3 are the central radii of inertia of the shell, s is the distance from the centre of mass of the shell to the centre of mass of the system, and $\delta = b_1/b_3$ is the aspect ratio of the cavity.

The components of the vectors $\boldsymbol{\upsilon}$, $\mathbf{e}$, $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ in the $Cx_1x_2x_3$ system will be denoted by υ_i, e_i, γ_i, ω_i and Ω_i ($i = 1, 2, 3$).

Following Zhukovsky, we introduce the notation for the inertia tensor of the transformed body $\mathbf{J}_* = \operatorname{diag}\{A_*, A_*, C_*\}$, which represents the sum of the inertia tensors of the shell ($\mathbf{J}_s$) and the equivalent body ($\mathbf{J}^*$).

We will write the equations of motion of the system in the central principal axes of inertia of the body using the theorem of the variation of the angular momentum about to the centre of mass C:

$$\mathbf{J}_*\dot{\boldsymbol{\omega}} + \mathbf{J}'\dot{\boldsymbol{\Omega}} + [\boldsymbol{\omega}, \mathbf{J}_*\boldsymbol{\omega} + \mathbf{J}'\boldsymbol{\Omega}] = [\mathbf{a}, -N\mathbf{e}] \tag{1.1}$$

To take into account the normal motion on the cavity wall, we will update the well-known Helmholtz equations for the uniform vortex motion of the fluid

$$\dot{\Omega}_1 = \omega_3\Omega_2 - (1-e)\omega_2\Omega_3 - e\Omega_2\Omega_3 = g_1$$
$$\dot{\Omega}_2 = -\omega_3\Omega_1 + (1-e)\omega_1\Omega_3 + e\Omega_1\Omega_3 = g_2$$
$$\dot{\Omega}_3 = (1+e)(\omega_2\Omega_1 - \omega_1\Omega_2) = g_3 \tag{1.2}$$

Here $e = (1-\delta^2)/(1+\delta^2)$ is a parameter that characterizes the aspect ratio of the cavity, and g_1, g_2 and g_3 are notations for the right-hand sides of the equations in system (1.2).

System (1.1), (1.2) can be reduced to the form

$$\dot{\mathbf{G}}_s + [\boldsymbol{\omega}, \mathbf{G}_s] = \mathbf{L}_p + \mathbf{M}, \quad \dot{\mathbf{G}}_l + [\boldsymbol{\omega}, \mathbf{G}_l] = -\mathbf{L}_p \tag{1.3}$$

Here

$$\mathbf{L}_p = (L_{p1}, L_{p2}, L_{p3})$$

$$L_{p1} = \frac{A_s}{A_*}\{A'\omega_3\Omega_2 - C'\omega_2\Omega_3 - A'g_1 + (B^* - C^* + B_s - C_s)\omega_2\omega_3\}$$

$$+ (C_s - A_s)\omega_2\omega_3 + \left(\frac{A_s}{A_*} - 1\right)M_1 \quad (123)$$

To obtain the remaining two components of the vector $\mathbf{L}_p$, cyclic transposition of the indices (123), as well as of the constants $(A'B'C')$, $(A_*B_*C_*)$, $(A^*B^*C^*)$, $(A_sB_sC_s)$ must be performed in the last equation. Here we assume that $A' = B'$, $A_* = B_*$, $A^* = B^*$, $A_s = B_s$; $M = (M_1, M_2, M_3)$ is the moment of the tension force of the rod.

To introduce the friction of the filler on the shell wall, we add the moment $\mathbf{L}_f$ to the right-hand side of the first equation in (1.3), and we add the moment $-\mathbf{L}_f$ to the right-hand side of the second equation. We obtain the system

$$\dot{\mathbf{G}}_s + [\boldsymbol{\omega}, \mathbf{G}_s] = \mathbf{L}_p + \mathbf{L}_f + \mathbf{M}, \quad \dot{\mathbf{G}}_l + [\boldsymbol{\omega}, \mathbf{G}_l] = -\mathbf{L}_p - \mathbf{L}_f \tag{1.4}$$

We will assume that the moment $\mathbf{L}_f$ depends linearly on the difference between the vortex vector of the filler and the angular velocity of the body:

$$-\mathbf{L}_f = \sigma(\boldsymbol{\Omega} - \boldsymbol{\omega}), \quad \sigma = \operatorname{diag}(\sigma_1, \sigma_1, \sigma_3), \quad \sigma_1 > 0, \quad \sigma_3 > 0$$

Thus, the coefficients σ_1 and σ_3 have been introduced phenomenologically.

We sum the left-hand and right-hand sides of the equations in system (1.4), and we replace this system with the equivalent system (a similar procedure was previously employed in Ref. 7):

$$\begin{aligned}
&A_*\dot{\omega}_1 + A'\dot{\Omega}_1 + (C_* - A_*)\omega_2\omega_3 + C'\omega_2\Omega_3 - A'\omega_3\Omega_2 = Ne_2a \\
&A_*\dot{\omega}_2 + A'\dot{\Omega}_2 + (A_* - C_*)\omega_3\omega_1 + A'\omega_3\Omega_1 - C'\omega_1\Omega_3 = -Ne_1a \\
&C_*\dot{\omega}_3 + C'\dot{\Omega}_3 + A'(\omega_1\Omega_2 - \omega_2\Omega_1) = 0
\end{aligned} \tag{1.5}$$

$$\begin{aligned}
&\dot{\Omega}_1 = \omega_3\Omega_3 - (1-e)\omega_2\Omega_3 - e\Omega_2\Omega_3 - \frac{\sigma_1 A_*}{A'A_s}(\Omega_1 - \omega_1) \\
&\dot{\Omega}_2 = -\omega_3\Omega_1 + (1-e)\omega_1\Omega_3 + e\Omega_1\Omega_3 - \frac{\sigma_1 A_*}{A'A_s}(\Omega_2 - \omega_2) \\
&\dot{\Omega}_3 = (1+e)(\omega_2\Omega_1 - \omega_1\Omega_2) - \frac{\sigma_3 C_*}{C'C_s}(\Omega_3 - \omega_3)
\end{aligned} \tag{1.6}$$

Thus, the mechanism of the interaction of the viscous filler with the shell walls is described by system of Eqs. (1.5), (1.6). We add to them the missing equations that describe the dynamics of the body on the rod, to them, and we write them in a single system of equations of motion relative to the principal axes of inertia of the system:

$$\begin{aligned}
&m\dot{\boldsymbol{\upsilon}} + [\boldsymbol{\omega}, m\boldsymbol{\upsilon}] = -mg\boldsymbol{\gamma} - N\mathbf{e}, \quad \mathbf{J}_*\dot{\boldsymbol{\omega}} + \mathbf{J}'\dot{\boldsymbol{\Omega}} + [\boldsymbol{\omega}, \mathbf{J}_*\boldsymbol{\omega} + \mathbf{J}'\boldsymbol{\Omega}] = [\mathbf{a}, -N\mathbf{e}] \\
&\dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega}, \boldsymbol{\gamma}] = 0, \quad l\dot{\mathbf{e}} + [\boldsymbol{\omega}, l\mathbf{e} - \mathbf{a}] = \boldsymbol{\upsilon}, \quad \mathbf{J}'\dot{\boldsymbol{\Omega}} + [(\boldsymbol{\omega} - \boldsymbol{\Omega}), \mathbf{L}\boldsymbol{\Omega}] = \mathbf{D}(\boldsymbol{\omega} - \boldsymbol{\Omega}), \quad (\mathbf{e}, \mathbf{e}) = 1
\end{aligned} \tag{1.7}$$

Here

$$\mathbf{D} = \operatorname{diag}\left\{\frac{\sigma_1 A_*}{A_s}, \frac{\sigma_1 A_*}{A_s}, \frac{\sigma_3 C_*}{C_s}\right\}, \quad \mathbf{L} = \operatorname{diag}\left\{A', A', \frac{4\delta^2}{(1+\delta^2)^2}C'\right\}$$

The tensor **D** characterizes the friction of the filler on the shell wall, and the tensor **L** has an auxiliary character.

The first equation in (1.7) expresses the law of variation of the momentum of the system, the second equation expresses the law of variation of the angular momentum of the system (this law is written down in scalar form in system (1.5)), the third equation is Poisson's equation, the fourth equation is the kinematic condition that connects the velocities of the points C and O of the rigid body, the fifth equation is the vector formulation of system (1.6) (when $\sigma = 0$, the tensor **D** vanishes, and the fifth equation is identical to the Helmholtz equation for describing the uniform vortex motion of a fluid), and the sixth equation is a relation that expresses the condition for the rod to be non-extensible: $|\overrightarrow{O_1O}|^2 = (l\mathbf{e}, l\mathbf{e}) = l^2 = \text{const}$.

In the chosen $Cx_1x_2x_3$ reference system, system of Eq. (1.7) is closed with respect to the unknown vectors $\boldsymbol{\upsilon}$, $\mathbf{e}$, $\boldsymbol{\gamma}$, $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ and the scalar N. In this reference system the vector $\mathbf{a}$ is constant and has the coordinates 0, 0, a.

2. The effective potential

System of Eq. (1.7) allows of the area integral

$$K = ([l\mathbf{e} - \mathbf{a}, m\boldsymbol{\upsilon}] + \mathbf{J}_*\boldsymbol{\omega} + \mathbf{J}'\boldsymbol{\Omega})\cdot\boldsymbol{\gamma} = k = \text{const} \tag{2.1}$$

and the geometric integral

$$\Gamma = (\boldsymbol{\gamma}, \boldsymbol{\gamma}) = 1 \tag{2.2}$$

The total mechanical energy of the system is a non-increasing function:

$$H = \frac{(m\boldsymbol{\upsilon}, \boldsymbol{\upsilon})}{2} + \frac{(\mathbf{J}_*\boldsymbol{\omega}, \boldsymbol{\omega})}{2} + \frac{(\mathbf{J}'\boldsymbol{\Omega}, \boldsymbol{\Omega})}{2} + mg(l\mathbf{e} - \mathbf{a}, \boldsymbol{\gamma});\ \frac{dH}{dt} = -(\mathbf{D}(\boldsymbol{\omega} - \boldsymbol{\Omega}), \boldsymbol{\omega} - \boldsymbol{\Omega}) \le 0 \tag{2.3}$$

Therefore, Routh's theory for dissipative systems with symmetry can be used to search for and investigate the steady motions of system (1.7).

We find the effective potential of the system as the minimum of the function H with respect to the variables $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$ and $\boldsymbol{\upsilon}$ at the fixed level of the first integral $K = k$:

$$W_k(\boldsymbol{\gamma}, \mathbf{e}) = \min_{\boldsymbol{\upsilon}, \boldsymbol{\omega}, \boldsymbol{\Omega}} H|_{K=k}$$

For this purpose, we introduce the function $F = H - \lambda(K - k)$, where λ is an undetermined Lagrange multiplier, and we write out the conditions for its stationary behaviour with respect to the variables $\boldsymbol{\omega}$, $\boldsymbol{\Omega}$, $\boldsymbol{\upsilon}$ and λ:

$$\frac{\partial F}{\partial \boldsymbol{\omega}} = \mathbf{J}_*(\boldsymbol{\omega} - \lambda\boldsymbol{\gamma}) = 0, \quad \frac{\partial F}{\partial \boldsymbol{\Omega}} = \mathbf{J}'(\boldsymbol{\Omega} - \lambda\boldsymbol{\gamma}) = 0$$
$$\frac{\partial F}{\partial \boldsymbol{\upsilon}} = m\boldsymbol{\upsilon} - \lambda m[\boldsymbol{\gamma}, l\mathbf{e} - \mathbf{a}] = 0, \quad \frac{\partial F}{\partial \lambda} = k - K = 0 \tag{2.4}$$

From the first three equalities in system (2.4), it follows that

$$\boldsymbol{\omega} = \lambda\boldsymbol{\gamma}, \quad \boldsymbol{\Omega} = \lambda\boldsymbol{\gamma}, \quad \boldsymbol{\upsilon} = \lambda[\boldsymbol{\gamma}, l\mathbf{e} - \mathbf{a}]$$

Substituting these values into the last equation of system (2.4), we find

$$\lambda = kJ^{-1}, \quad J = ((\mathbf{J}_* + \mathbf{J}')\boldsymbol{\gamma}, \boldsymbol{\gamma}) + m[\boldsymbol{\gamma} \times (l\mathbf{e} - \mathbf{a})]^2$$

Thus, a minimum of the function H at the level $K = k$ is reached for

$$\boldsymbol{\omega} = kJ^{-1}\boldsymbol{\gamma}, \quad \boldsymbol{\Omega} = kJ^{-1}\boldsymbol{\gamma}, \quad \boldsymbol{\upsilon} = kJ^{-1}[\boldsymbol{\gamma}, l\mathbf{e} - \mathbf{a}] \tag{2.5}$$

and is equal to

$$W_k(\boldsymbol{\gamma}, \mathbf{e}) = k^2J^{-1}/2 + \Pi; \quad \Pi = mg(l\mathbf{e} - \mathbf{a}, \gamma) \tag{2.6}$$

To make the mathematical expressions more concise in the ensuing discussion, we introduce the following notation for the components of the tensor $\mathbf{J}_* + \mathbf{J}'$:

$$A = A_* + A', \quad C = C_* + C'$$

3. The general properties of steady motions

According to Routh's theory, the critical points

$$\boldsymbol{\gamma} = \boldsymbol{\gamma}_0, \quad \mathbf{e} = \mathbf{e}_0 \tag{3.1}$$

of the function $W_k(\boldsymbol{\gamma}, \mathbf{e})$ on the manifold $\{\boldsymbol{\gamma}^2 = 1, \mathbf{e}^2 = 1\}$ correspond to the steady motions described by Eqs. (2.5) and (3.1), which are clearly permanent rotations of the system as a rigid body ($\boldsymbol{\omega} = \boldsymbol{\Omega}$) about the vertical, and the minimum points correspond to stable steady motions.

Assertion 1. In any steady motion described by Eqs. (2.5) and (3.1), the vectors $\mathbf{a}$, $\mathbf{e}$ and $\boldsymbol{\gamma}$ lie in one plane.

Proof. After scalar multiplication of the second equation in (1.7) by ω, we obtain the equality

$$(\mathbf{J}_*\dot{\boldsymbol{\omega}}, \boldsymbol{\omega}) + (\mathbf{J}'\dot{\boldsymbol{\Omega}}, \boldsymbol{\omega}) = N([\mathbf{e}, \mathbf{a}], \boldsymbol{\omega}) \tag{3.2}$$

from which, for the steady motions described by (2.5) and (3.1), we have

$$NkJ^{-1}([\mathbf{e}, \mathbf{a}], \boldsymbol{\gamma}) \equiv 0$$

Since the multiplier $NkJ^{-1} \neq 0$ in the general case, the mixed product of the vectors $\mathbf{a}$, $\mathbf{e}$ and $\boldsymbol{\gamma}$ is equal to zero. This means that these vectors are coplanar.

Assertion 2. The energy function H decreases in all motions that differ from the steady motions described by Eqs. (2.5) and (3.1).

Proof. According to formulae (2.3), $dH/dt \equiv 0$ if and only if $\boldsymbol{\omega} = \boldsymbol{\Omega}$. It follows from the fifth equation of (1.7) that $\dot{\omega} = \dot{\Omega} \equiv 0$. Substituting these relations into equality (3.2), we obtain

$$N([\mathbf{e}, \mathbf{a}], \boldsymbol{\omega}) = N([\mathbf{a}, \boldsymbol{\omega}], \mathbf{e}) \equiv 0 \tag{3.3}$$

Since $N \neq 0$ in the general case, the following two cases are possible:

a) $\mathbf{a} = \mu\boldsymbol{\omega}$, where μ is a constant;
b) the vectors $\mathbf{a}$ and $\boldsymbol{\omega}$ are not collinear.

In the second case,

$$\mathbf{e} = \mu\mathbf{i}_3 + \lambda\frac{\boldsymbol{\omega}}{\omega}; \quad \boldsymbol{\omega} = \omega_1\mathbf{i}_1 + \omega_2\mathbf{i}_2 + \omega_3\mathbf{i}_3, \quad \omega = \sqrt{\omega_1^2 + \omega_2^2 + \omega_3^2} \neq 0$$

where $\mu = \mu(t)$ and $\lambda = \lambda(t)$ are functions of time t. If $\omega = 0$, the system is in an equilibrium position, which is a special case of permanent rotation.

Case a. Substituting the expression $\mathbf{a} = \mu\boldsymbol{\omega}$ into the second equation of (1.7), we obtain

$$[\boldsymbol{\omega}, (\mathbf{J}_* + \mathbf{J}')\boldsymbol{\omega} + N\mathbf{e}] \equiv 0 \Rightarrow (\mathbf{J}_* + \mathbf{J}')\boldsymbol{\omega} + N\mathbf{e} = \nu\boldsymbol{\omega}$$

where ν is a certain constant. Differentiation of this equation gives the equality

$$\dot{N}\mathbf{e} + N\dot{\mathbf{e}} = 0$$

Multiplying this equality sealarly by the vector $\mathbf{e}$ and taking into account the last equality of (1.7), we obtain $\dot{N} \equiv 0$. Therefore, $\dot{\mathbf{e}} \equiv 0$. In addition, it follows from the fourth equation of (1.7) that the velocity vector $\boldsymbol{\upsilon}$ is also constant, and it follows from the first equation that $\dot{\boldsymbol{\gamma}} \equiv 0$. Finally, according to the third equation of (17), we have a linear relation between the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$.

Case b. We have

$$\mathbf{e} = \frac{\lambda\omega_1}{\omega}\mathbf{i}_1 + \frac{\lambda\omega_2}{\omega}\mathbf{i}_2 + \left(\frac{\lambda\omega_3}{\omega} + \mu\right)\mathbf{i}_3, \quad \dot{\mathbf{e}} = \frac{\dot{\lambda}\omega_1}{\omega}\mathbf{i}_1 + \frac{\dot{\lambda}\omega_2}{\omega}\mathbf{i}_2 + \left(\frac{\dot{\lambda}\omega_3}{\omega} + \dot{\mu}\right)\mathbf{i}_3 \tag{3.4}$$

Since $|\mathbf{e}| = 1$, λ and μ are related by the following expression

$$\mu^2 + 2\lambda\mu\frac{\omega_3}{\omega} + \lambda^2 = 1 \tag{3.5}$$

It follows form the second equation of (1.7) (or Eq. (1.5) that

$$N = \frac{(C - A)\omega\omega_3}{a\lambda} \tag{3.6}$$

In this formula it is assumed that $\lambda \neq 0$. If $\lambda = 0$, then $\mathbf{e} = \mu\mathbf{i}_3$, $(\mathbf{e}, \dot{\mathbf{e}}) = \mu\dot{\mu} = 0$. Hence it follows that $\dot{\mu} = 0$ and $\mathbf{e} = \text{const}$. This case corresponds to a trivial uniform rotation, i.e., a rotation during which the axis of symmetry of the body and the rod lie on a vertical.

Substituting expressions (3.4) into the fourth equation of (1.7), we obtain

$$\boldsymbol{\upsilon} = l\left[\frac{\dot{\lambda}\omega_1}{\omega} + \omega_2\left(\mu - \frac{a}{l}\right)\right]\mathbf{i}_1 + l\left[\frac{\dot{\lambda}\omega_2}{\omega} - \omega_1\left(\mu - \frac{a}{l}\right)\right]\mathbf{i}_2 + l\left[\frac{\dot{\lambda}\omega_3}{\omega} + \dot{\mu}\right]\mathbf{i}_3$$

$$\dot{\boldsymbol{\upsilon}} = l\left[\frac{\ddot{\lambda}\omega_1}{\omega} + \omega_2\dot{\mu}\right]\mathbf{i}_1 + l\left[\frac{\ddot{\lambda}\omega_2}{\omega} - \omega_1\dot{\mu}\right]\mathbf{i}_2 + l\left[\frac{\ddot{\lambda}\omega_3}{\omega} + \ddot{\mu}\right]\mathbf{i}_3 \tag{3.7}$$

Finding an expression for the unit vector of an ascending vertical from the first equation in system (1.7) and using the formulae for N, $\mathbf{e}$, $\boldsymbol{\upsilon}$ and $\dot{\boldsymbol{\upsilon}}$, we obtain for the components of the unit vector $\boldsymbol{\gamma}$

$$\gamma_1 = -\frac{(C - A)\omega_1\omega_3}{mga} - \frac{l}{g}\left(\ddot{\lambda}\frac{\omega_1}{\omega} + \omega_1\omega_3\left(\mu - \frac{a}{l}\right) + 2\omega_2\dot{\mu}\right)$$

$$\gamma_2 = -\frac{(C - A)\omega_2\omega_3}{mga} - \frac{l}{g}\left(\ddot{\lambda}\frac{\omega_2}{\omega} + \omega_2\omega_3\left(\mu - \frac{a}{l}\right) - 2\omega_1\dot{\mu}\right)$$

$$\gamma_3 = -\frac{(C - A)\omega_3}{mga}\left(\omega_3 + \omega\frac{\mu}{\lambda}\right) - \frac{l}{g}\left(\ddot{\lambda}\frac{\omega_3}{\omega} + \ddot{\mu} + (\omega_1^2 + \omega_2^2)\left(\mu - \frac{a}{l}\right)\right) \tag{3.8}$$

In addition, Poisson's equation (the third equation of (1.7)) holds for the unit vector $\boldsymbol{\gamma}$. In projections onto the axes of the $Cx_1x_2x_3$ system, this equation gives

$$\dot{\gamma}_1 + \omega_2\gamma_3 - \omega_3\gamma_2 = 0 \tag{3.9}$$

$$\dot{\gamma}_2 + \omega_3\gamma_1 - \omega_1\gamma_3 = 0 \tag{3.10}$$

$$\dot{\gamma}_3 + \omega_1\gamma_2 - \omega_2\gamma_1 = 0 \tag{3.11}$$

We will write Eqs. (3.9) and (3.10) in an equivalent form. For this purpose, we first multiply Eq. (3.9) by ω_1 and Eq. (3.10) by ω_2, and add them. We then multiply Eq. (3.9) by ω_2 and Eq. (3.10) by $-\omega_1$, and we also add them. As a result, we obtain

$$\omega_1\dot{\gamma}_1 + \omega_2\dot{\gamma}_2 - \omega_3(\omega_2\gamma_1 - \omega_1\gamma_2) = 0 \tag{3.12}$$

$$\omega_2\dot{\gamma}_1 - \omega_1\dot{\gamma}_2 - \omega_3(\omega_1\gamma_1 + \omega_2\gamma_2) + \gamma_3(\omega_1^2 + \omega_2^2) = 0 \tag{3.13}$$

In changing from system (3.9)–(3.11) to system (3.11)–(3.13), we assume that $\omega_1^2 + \omega_2^2 \neq 0$. If $\omega_1 = \omega_2 = 0$, then $\mathbf{e} = (\lambda\omega_3/\omega + \mu)\mathbf{i}_3$, and $(\mathbf{e}, \dot{\mathbf{e}}) = (\lambda\omega_3/\omega + \mu)(\dot{\lambda}\omega_3/\omega + \dot{\mu}) = 0$. Since the first multiplier cannot be equal to zero, we have $(\dot{\lambda}\omega_3/\omega + \dot{\mu}) = 0$ and $\mathbf{e} = \text{const}$. This case corresponds to a trivial uniform rotation.

After substituting the first two expressions in (3.8) into Eq. (3.11), we obtain

$$\gamma_3 = \frac{2l}{g}(c_1 - (\omega_1^2 + \omega_2^2)\mu) \tag{3.14}$$

In a similar manner, from Eqs. (3.12) and (3.13) we obtain

$$\ddot{\lambda} = \omega_3\omega\mu + c_2 \tag{3.15}$$

Here c_1 and c_2 are arbitrary constants of integration

Taking into account equalities (3.14) and (3.15), we write the expression for $\ddot{\mu}$ in the form

$$\ddot{\mu} = \mu(\omega_3^2 - \omega_1^2 - \omega_2^2) + c_1 + \frac{\omega_3}{2\omega}c_2 + \frac{(C-A)\omega_3^2}{2mal} - \frac{\omega_3^2 a}{2l} \tag{3.16}$$

Substituting expressions (3.14)–(3.16) into the last equality of (3.8), we obtain an algebraic equation in λ and μ with constant coefficients:

$$\mu(\omega_1^2 + \omega_2^2 - \omega_3^2) - \frac{(C-A)\omega_3\omega}{2mal}\frac{\mu}{\lambda} - \frac{3(C-A)\omega_3^2}{4mal} + \\ + \frac{a}{4l}(2\omega_1^2 + 2\omega_2^2 - \omega_3^2) + \frac{3c_1}{2} - \frac{3\omega_3 c_2}{4\omega} = 0 \tag{3.17}$$

It follows from Eqs. (3.5) and (3.17) that λ and μ are constant. Therefore, the vector $\mathbf{e}$ and the scalar N are also constant (see relations (3.4) and (3.6)). It follows from the fourth equation of (1.7) that the velocity $\boldsymbol{v}$ is constant, and it follows from the first equation of (1.7) that the unit vector of the ascending vertical $\boldsymbol{\gamma}$ is constant. Finally it follows from the third equation of (1.7) that the vectors $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are linearly dependent. Thus, Assertion 2 has been proved.

Hence, according to the theorem of partial asymptotic stability for dissipative systems[8], it follows that isolated minimum points of the effective potential correspond to asymptotically partially stable motions, and the other isolated critical points correspond to unstable motions. In the former case, partial asymptotic stability means that a perturbed motion tends to a permanent rotation (but not necessarily to an unperturbed permanent rotation).

4. The equations of steady motions

We will investigate the steady motions of the dynamical system under consideration in coordinates that are not connected to the central principal axes of inertia of the body. At the fixed point O_1 we introduce the system of coordinates $O_1y_1y_2y_3$, which rotates with angular velocity $\eta = kJ^{-1}$ about its O_1y_3 axis, which is directed vertically upwards. The other two axes were chosen so that the unit vectors of the axes of the system of coordinates would comprise a right-hand orthonormalized reference system.

The equations of the steady motions will have the form

$$\delta W_k = \delta\left(\frac{1}{2}k^2J^{-1} + \Pi\right) = -\frac{k^2}{2J^2}\delta J + \delta\Pi = -\frac{1}{2}\eta^2\delta J + \delta\Pi = 0$$

Therefore, to find the steady motions we can use the equations of relative equilibria in the $O_1y_1y_2y_3$ system of coordinates that rotates with a constant angular velocity $\eta = \text{const}$ (see also Ref. 1):

$$\delta W_\eta = 0; \quad W_\eta = -\frac{1}{2}\eta^2 J + \Pi$$

Following a well-known approach,[6] we introduce the scalars e_{y_j} and use i_{1j}, i_{2j}, i_{3j} to denote the projections of the vectors $\mathbf{e}$, $\mathbf{i}_1$, $\mathbf{i}_2$ and $\mathbf{i}_3$ onto the O_1y_j axis (j = 1, 2, 3). The following geometrical relations hold

$$\pi_{e_y} = e_{y_1}^2 + e_{y_2}^2 + e_{y_3}^2 - 1 = 0, \quad \pi_{i_r} = i_{r1}^2 + i_{r2}^2 + i_{r3}^2 - 1 = 0$$
$$\pi_{i_si_t} = i_{s1}i_{t1} + i_{s2}i_{t2} + i_{s3}i_{t3} = 0; \quad r, s, t = 1, 2, 3; \quad s \neq t \tag{4.1}$$

In the $O_1y_1y_2y_3$ system of coordinates we have

$$\Pi = mg(le_{y_3} - ai_{33}), \quad J = A(i_{13}^2 + i_{23}^2) + Ci_{33}^2 + m[(le_{y_1} - ai_{31})^2 + (le_{y_2} - ai_{32})^2]$$

Henceforth, instead of W_η we will consider the function

$$W_* = W_\eta + \frac{1}{2}\eta^2\left(\lambda_{e_y}\pi_{e_y} + \sum_{r=1}^{3}\lambda_{i_r}\pi_{i_r} + \sum_{s,t=1,s\neq t}^{3}\lambda_{i_si_t}\pi_{i_si_t}\right)$$

where λ_{e_y}, λ_{i_r}, $\lambda_{i_si_t}$ are undetermined Lagrange multipliers.

We will seek the steady motions described by Eqs. (2.5) and (3.1) in the system under consideration, based on the conditions for steady behaviour of the function W_* with respect to the variables e_{y_j} and i_{rs}

$$\delta W_* = 0 \tag{4.2}$$

taking relations (4.1) into account. We have

$$\frac{\partial W_*}{\partial i_{\alpha\beta}} = \eta^2(\lambda_{i_\alpha}i_{\alpha\beta} + \lambda_{i_1i_2}i_{3-\alpha\beta]} + \lambda_{i_\alpha i_3}i_{3\beta}) = 0$$
$$\frac{\partial W_*}{\partial i_{\alpha 3}} = \eta^2((\lambda_{i_\alpha} - A)i_{\alpha 3} + \lambda_{i_1i_2}i_{3-\alpha 3} + \lambda_{i_\alpha i_3}i_{33}) = 0, \quad \alpha, \beta = 1, 2 \tag{4.3}$$
$$\frac{\partial W_*}{\partial i_{3\alpha}} = \eta^2(\lambda_{i_3}i_{3\alpha} + \lambda_{i_1i_3}i_{1\alpha} + \lambda_{i_2i_3}i_{2\alpha} + ma(le_{y_\alpha} - ai_{3\alpha})) = 0, \quad \alpha = 1, 2$$
$$\frac{\partial W_*}{\partial i_{33}} = \eta^2\left((\lambda_{i_3} - C)i_{33} + \lambda_{i_1i_3}i_{13} + \lambda_{i_2i_3}i_{23} - \frac{mga}{\eta^2}\right) = 0 \tag{4.4}$$
$$\frac{\partial W_*}{\partial e_{y_\alpha}} = \eta^2((\lambda_{e_y} - ml^2)e_{y_\alpha} + mali_{3\alpha}) = 0, \ \alpha = 1, 2; \ \frac{\partial W_*}{\partial e_{y_3}} = \eta^2\lambda_{e_y}e_{y_3} + mgl = 0 \tag{4.5}$$

From (4.5) we obtain

$$e_{y_\alpha} = -\frac{mal}{\lambda_{e_y}^*}i_{3\alpha}, \quad \alpha = 1, 2, \quad e_{y_3} = -\frac{ml^2}{\lambda_{e_y}w}; \quad \lambda_{e_y}^* = \lambda_{e_y} - ml^2, \quad w = \frac{\eta^2 l}{g} \tag{4.6}$$

By virtue of the fact that the expressions for e_{y_1} and e_{y_2} have the identical multiplier $mal/\lambda_{e_y}^*$, it also follows that in any steady motions, the vectors $\mathbf{e}$, $\mathbf{i}_3$ and $\boldsymbol{\gamma}$ lie in the same plane, since their mixed product equals zero.

Next, using conditions (4.1), from equalities (4.3) we obtain

$$\lambda_{i_\alpha} = Ai_{\alpha 3}^2, \quad \lambda_{i_\alpha i_\gamma} = Ai_{\alpha 3}i_{\gamma 3}, \quad \alpha = 1, 2, \quad \gamma = 2, 3 \quad (\alpha \neq \gamma) \tag{4.7}$$

On the other hand, using relations (4.6), from equalities (4.4) we obtain

$$\lambda_{i_\alpha i_3} - Ci_{\alpha 3}i_{33} - \left[\frac{(mal)^2}{\lambda_{e_y}^*} + ma^2\right](i_{31}i_{\alpha 1} + i_{32}i_{\alpha 2}) - \frac{mal}{w}i_{\alpha 3} = 0, \quad \alpha = 1, 2$$

$$\lambda_{i_3} = Ci_{33}^2 + \left[\frac{(mal)^2}{\lambda_{e_y}^*} + ma^2\right](1 - i_{33}^2) + \frac{mal}{w}i_{33}$$

Substituting expression (4.7) for $\lambda_{i_\alpha i_3}$ here, we have

$$(i_{33} - i_{33}^*)i_{\alpha 3} = 0, \quad \alpha = 1, 2; \quad i_{33}^* = \frac{mal\lambda_{e_y}^*}{w[(A - C + ma^2)\lambda_{e_y}^* + (mal)^2]} \tag{4.8}$$

Hence it follows that only two cases are possible.

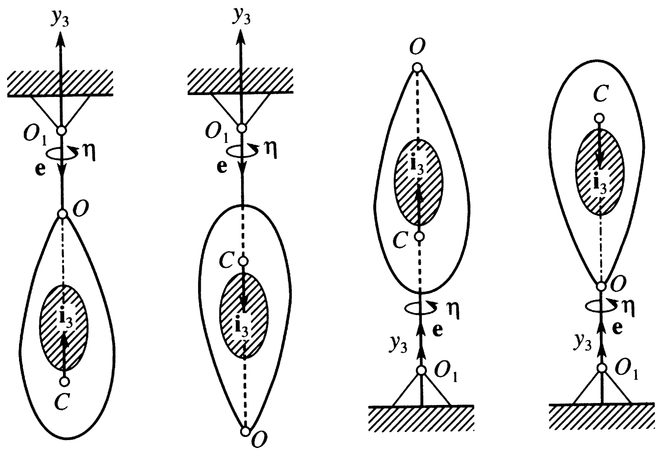

Fig. 2.

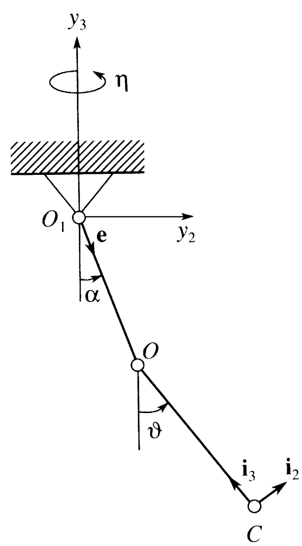

Fig. 3.

The case when $i_{33} \neq i_{33}^*$. In this case $i_{13} = i_{23} = 0$ simultaneously, and $i_{33}^2 = 1$. This case corresponds to families of trivial steady motions, i.e., uniform rotations, for which the points O_1, O and C form a straight line coinciding with the vertical. These motions exist for any value of w, i.e., for any value of the constant area integral k. They are shown in Fig. 2. The following notation is introduced for these motions

$$S^{-+}(e_{y_3} = -1, i_{33} = 1), \quad S^{--}(e_{y_3} = -1, i_{33} = -1), \quad S^{++}(e_{y_3} = 1, i_{33} = 1),$$
$$S^{+-}(e_{y_3} = 1, i_{33} = -1) \tag{4.9}$$

The case when $i_{33} = i_{33}^*$. The quantities i_{13} and i_{23} can take any value. This case corresponds to families of "oblique" uniform rotations (Fig. 3). They do not exist for any value of w (any value of k), since $|i_{33}| < 1$ (the case when $|i_{33}| = 1$ corresponds to the case when $i_{33} \neq i_{33}^*$.

It was shown above that the only steady motions in this problem are permanent rotations of the system as a whole about the vertical. Since the body is dynamically symmetrical, the $Cx_1x_2x_3$ system of coordinates can be turned relative to the $O_1y_1y_2y_3$ system in such a manner that the vector $\mathbf{i}_1$ will always be parallel to the O_1y_1 axis without a loss of generality (see Fig. 3). For this reason, we set $e_{y_1} = 0, i_{11} = 1, i_{12} = i_{13} = 0, i_{21} = i_{31} = 0$ in any permanent rotation.

5. Trivial steady motions

It follows from the equations of the steady motions that there are four one-parameter families of trivial steady motions (4.9), in which

$$e_{y_1} = e_{y_2} = 0, \quad i_{11} = i_{22} = 1, \quad i_{12} = i_{13} = i_{21} = i_{23} = i_{31} = i_{32} = 0$$
$$k^2 = w\left(\frac{Cg}{l}\right)^2, \quad \lambda_{i_3} = C + \frac{mal}{w} i_{33}, \quad \lambda_{e_y} = -\frac{ml^2}{e_{y_3} w}, \quad \lambda_{i_1} = \lambda_{i_2} = \lambda_{i_1 i_2} = \lambda_{i_1 i_3} = \lambda_{i_2 i_3} = 0 \tag{5.1}$$

Table 1

1) Solution	S^{-+}			S^{+-}	
2) Degree of instability	0	2	4	2	4
$p>0$	$w<w_-$	$w\in(w_-,w_+)$	$w>w_+$	–	$w\in\mathbb{R}_+$
$p<0$	$w<w_-$	$w>w_-$	–	$w>w_-$	$w<w_-$

To investigate their stability, we calculate the second variation $\delta^2 W_*$ of the function W_* on the manifold

$$M = \{\delta\pi_{e_y} = 0, \delta\pi_{i_1} = 0, \delta\pi_{i_2} = 0, \delta\pi_{i_3} = 0, \delta\pi_{i_1 i_2} = 0, \delta\pi_{i_1 i_3} = 0, \delta\pi_{i_2 i_3} = 0\} \quad (5.2)$$

For each of the trivial steady motions, the second variation $\delta^2 W_*$ is the sum of two identical quadratic forms:

$$\delta^2 W_* = F_1 + F_2 \quad (5.3)$$

where

$$F_\alpha = -\left[mglw + \frac{mgl}{e_{y_3}}\right](\delta e_{y_\alpha})^2 + 2wmga\delta e_{y_\alpha}\delta i_{3\alpha} + \left[(C - A - ma^2)\frac{wg}{l} + ngai_{33}\right](\delta i_{3\alpha})^2$$

$$\alpha = 1, 2$$

Therefore, the degree of instability can take the values 0, 2 and 4 only. (For the solutions S^{++} and S^{+-} the degree of instability can take the values 2 and 4 only, since the coefficient in front of $(\delta e_{y_\alpha})^2$ is negative.)

We will use Δ_1 and Δ_2 to denote the first- and second-order minors, respectively, of the quadratic form F_α, and, for convenience, we will introduce the two dimensionless parameters

$$p = (A - C)/(ma^2), \quad q = l/a$$

which depend on the inertial and geometrical characteristics of the system.

Let us examine the solution $S^{-+}(e_{y_3} = -1, i_{33} = 1)$. It corresponds to a trivial permanent rotation of the system, during which the centre of mass C lies below the suspension point O, and the suspension point O lies below the fixed point O_1. The stability conditions of the solution are written in the form of the system of inequalities

$$-w + 1 > 0, \quad w^2 p - w(p + 1 + q) + q > 0 \quad (5.4)$$

From the first inequality, we find that

$$\Delta_1 > 0 \text{ for } w < 1 \ (\eta^2 < \eta_*^2 = g/l)$$

The discriminant of the trinomial that is quadratic in w from the second inequality of (5.4) is positive:

$$D = (p + 1 + q)^2 - 4pq = (p + 1 - q)^2 + 4q > 0$$

Therefore, there are two real roots

$$w_\pm = (p + 1 + q \pm \sqrt{D})/(2p) \quad (5.5)$$

It can be shown that the condition

$$0 < w_- < 1 < w_+$$

holds for these roots when $p > 0$ (an elongated body) and that the condition

$$w_+ < 0 < w_- < 1$$

holds when $p < 0$ (a flattened body). In the latter case, the root w_+ should clearly be discarded.

Applying Sylvester's criteria to the quadratic form $\delta^2 W_*$, we can obtain the stability conditions of the solution S^{-+}: the results are shown in the left-hand columns of Table 1. Similar calculations can be performed for S^{--} (S^{++}), which corresponds to a rotation of the system during which the centre of mass lies above (below) the suspension point and the suspension point is below (above) the fixed point. The results are shown in the left-hand (right-hand) columns of Table 2. The results for the solution S^{+-}, which corresponds to a rotation of the system during which the centre of mass is located above the suspension point, and the suspension point is above the fixed point, are shown in the right-hand columns of Table 1.

6. General properties of non-trivial steady motions

As was shown in Section 4, if $i_{13}^2 + i_{23}^2 \neq 0$, we have the case of "oblique" steady motions. They are specified by the following relations

$$e_{y_1} = 0, \quad i_{11} = 1, \quad i_{12} = i_{13} = i_{21} = i_{31} = 0 \quad (6.1)$$

Table 2

1) Solution	S^{--}			S^{++}	
2) Degree of instability	0	2	4	2	4
$p>0$	–	$w<w_+$	$w>w_+$	$w<w_+$	$w>w_+$
$p<p^+$	$w\in(w_+, w_-)$	$w\in U$	–	$w\in\mathbb{R}_+$	–
$p^+<p<p^-$	–	$w\in\mathbb{R}_+$	–	$w\in\mathbb{R}_+$	–
$p^-<p<0, q<1$	–	$w\in\mathbb{R}_+$	–	$w\in U$	$w\in(w_+, w_-)$
$p^-<p<0, q>1$	$w\in(w_+, w_-)$	$w\in U$	–	$w\in\mathbb{R}_+$	–

3) Note: $p^{\pm}=-(1\pm\sqrt{q})^2$, $U=\{w\in\mathbb{R}_+ : w<w_+, w>w_-\}$.

$$e_{y_2}=\frac{e_{y_3}w\xi}{q}i_{32},\quad e_{y_2}^2+e_{y_3}^2=1,\quad i_{22}=i_{33}=\frac{q}{w(p+\xi)},\quad i_{23}=-i_{32},\quad i_{32}^2+i_{33}^2=1 \tag{6.2}$$

$$k^2(e_{y_3}, i_{33}, w)=\frac{J^2wg}{l},\quad J=C+ma^2(p+\xi^2)(1-i_{33}^2);\quad \xi=\frac{1}{1+e_{y_3}w} \tag{6.3}$$

Here it is assumed that the $Cx_1x_2x_3$ system of coordinates is turned relative to the $O_1y_1y_2y_3$ system so that the Cx_1 and O_1y_1 axes would be directed in the same direction; therefore it may be assumed that relations (6.1) hold for steady motions.

We express relations (6.2) in terms of the variables e_{y_3} and i_{33} and the dimensionless parameter w:

$$\Psi(e_{y_3}, i_{33}, w)=i_{33}-\frac{q}{w(p+\xi)}=0,\ \ \Phi(e_{y_3}, i_{33}, w)=-1+e_{y_3}^2+\frac{(e_{y_3}w\xi)^2}{q^2}(1-i_{33}^2)=0 \tag{6.4}$$

Next, from the second relation of (6.4), we express w in terms of e_{y_3} and i_{33}. We have

$$\frac{e_{y_3}w}{q(1-e_{y_3}w)}=\pm\sqrt{c(e_{y_3}, i_{33})};\quad c=c(e_{y_3}, i_{33})=\frac{1-e_{y_3}^2}{1-i_{33}^2} \tag{6.5}$$

The expression under the radical sign is always positive by virtue of the constraints imposed on e_{y_3} and i_{33}.

Consider Eq. (6.5) when the plus sign is chosen. We will denote the value of the parameter w that satisfies this equation by w_1. We obtain

$$w_1(e_{y_3}, i_{33})=\frac{q\sqrt{c}}{(1-q\sqrt{c})e_{y_3}}\left(\sqrt{c}\neq\frac{1}{q}\right) \tag{6.6}$$

Since w can only take positive values, w_1 exists only when $(1-q\sqrt{c})e_{y_3}>0$.

Consider Eq. (6.5) with the minus sign. We will denote the value of the parameter w that satisfies this equation by w_2. We obtain

$$w_2(e_{y_3}, i_{33})=-\frac{q\sqrt{c}}{(1+q\sqrt{c})e_{y_3}} \tag{6.7}$$

In this equation w_2 exists only when $e_{y_3}<0$.

We will introduce the following notation for the surfaces

$$F_\alpha=\{(e_{y_3}, i_{33}, k^2) : k^2-k^2(e_{y_3}, i_{33}, w_\alpha)=0\}$$

$$\Psi_\alpha=\{(e_{y_3}, i_{33}, k^2) : \Psi(e_{y_3}, i_{33}, w_\alpha)=0\};\quad \alpha=1, 2$$

and for regions in the space e_{y_3}, i_{33}, k^2

$$G_1=\left\{(e_{y_3}, i_{33}, k^2) : e_{y_3}<0,\ e_{y_3}^2-\frac{i_{33}^2}{q^2}<1-\frac{1}{q^2}\right\}$$

$$G_2=\left\{(e_{y_3}, i_{33}, k^2) : e_{y_3}<0,\ e_{y_3}^2-\frac{i_{33}^2}{q^2}>1-\frac{1}{q^2}\right\}$$

$$G_3=\left\{(e_{y_3}, i_{33}, k^2) : e_{y_3}>0,\ e_{y_3}^2-\frac{i_{33}^2}{q^2}>1-\frac{1}{q^2}\right\}$$

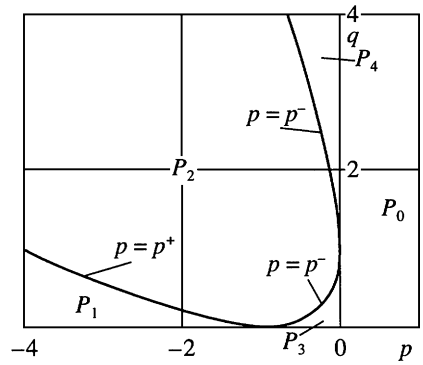

Fig. 4.

and we summarize the results obtained above.

Region	G_1	G_2	G_3
Rotation frequency	w_1 w_2	w_2	w_1

Therefore, the oblique steady motions can be represented as the union of the two independent intersections $\Psi_1 \cap F_1$ and $\Psi_2 \cap F_2$ in the region G_1 (which corresponds to the solutions S^{-+} and S^{--}), as the intersection $\Psi_2 \cap F_2$ in the region G_2 (which corresponds to the solutions S^{-+} and S^{--}), and as the intersection $\Psi_1 \cap F_1$ in region G_3 (which corresponds to the solutions S^{++} and S^{+-}).

7. Initial portions of non-trivial steady motions in the vicinity of trivial motions

According to bifurcation theory, non-trivial ("oblique") steady motions should branch off at points where the degree of instability of trivial permanent rotations changes.

We will treat the half-plane $P = \{(p, q) : p \in \mathbb{R}, q > 0\}$ as the set of all physically possible values of the parameters p and q of the system. (Note that a natural constraint exists for p by virtue of the relations between the moments of inertia of the system: $p \geq A/(ma^2)$.) In Fig. 4 this half-plane is divided by the coordinate axes and plots of $p = p^{\pm}$ into the following regions

$$P_0 = \{(p, q) : p > 0\}, \quad P_1 = \{(p, q) : p < p^+\}, \quad P_2 = \{(p, q) : p^+ < p < p^-\}$$

$$P_3 = \{(p, q) : p^- < p < 0, q < 1\}, \; P_4 = \{(p, q) : p^- < p < 0, q > 1\}; \; p^{\pm} = -(1 \pm \sqrt{q})^2$$

According to the data in Tables 1 and 2, in each of these regions, all four families of trivial steady motions have a fixed number of points where the degree of instability changes.

We will investigate the behaviour of the one-parameter families of non-trivial steady motions in the neighbourhoods of the bifurcation points using the small-parameter method. Since it has previously been proved that the vectors **e**, $\mathbf{i}_3$ and $\boldsymbol{\gamma}$ for any steady motions lie in one plane, it is sufficient to assign two angles to describe the orientation of the rod and the axis of symmetry of the body in any non-trivial permanent rotation. Let α be the angle between the vectors $-\boldsymbol{\gamma}$ and **e**, and, therefore, $e_{y_3} = -\cos\alpha$. Also, let ϑ be the angle between the vectors $\boldsymbol{\gamma}$ and $\mathbf{i}_3$, and, therefore, $i_{33} = \cos\vartheta$. The trivial steady motions correspond to the following values of α and ϑ

Solution	S^{-+}	S^{--}	S^{++}	S^{+-}
α	0	0	π	π
ϑ	0	π	0	π

Consider the points where the degree of instability changes for the solution S^{-+}. According to the data in Table 1, for $p > 0$ there are two bifurcation points, which correspond to the bifurcation values $w_{\pm} = \eta_{\pm}^2 l/g$; for $p < 0$ there is one point, which corresponds to the bifurcation value $w_{-} = \eta_{-} l/g$. In the neighbourhoods of these points, the angles α and ϑ are small. Let α be a small parameter. Following the general technique of investigating bifurcations by the small-parameter method, we express ϑ and w in terms of α:

$$\vartheta = x_{\pm}\alpha + o(\alpha), \quad w = w_{\pm} + z_{\pm}\alpha^2 + o(\alpha^2) \tag{7.1}$$

where $x_{\pm}$ and $z_{\pm}$ are constants that depend on the parameters of the system.

Since

$$e_{y_3} = -\cos\alpha = -1 + \alpha^2/2 + o(\alpha^2), \quad i_{33} = \cos\vartheta = 1 - \vartheta^2/2 + o(\vartheta^2)$$

after substituting these expressions into Eq. (6.4) taking (7.1) into account, retaining terms containing powers of α up to the second inclusive, and equating the coefficients of like powers of α, we obtain

$$-q^2(w_{\pm} - 1)^2 + w_{\pm}^2 x_{\pm}^2 = 0 \tag{7.2}$$

$$pw_\pm^2 - w_\pm(p+1+q) + q = 0 \tag{7.3}$$

$$-x_\pm^2 p^2 w_\pm^4 + 2x_\pm^2 p(p+1)w_\pm^3 + (2qpz_\pm - x_\pm^2(p+1)^2 - q)w_\pm^2 - 4qz_\pm p w_\pm + 2qz_\pm(p+1) = 0 \tag{7.4}$$

From Eq. (7.2) we find

$$x_\pm^2 = q^2(w_\pm - 1)^2/w_\pm^2 \tag{7.5}$$

Eq. (7.3) is satisfied identically by virtue of relation (5.5). Substituting expression (7.5) into Eq. (7.4), we obtain

$$z_\pm = [qp^2w_\pm^4 - 2qp(2p+1)w_\pm^3 + (6qp^2 + 6qp + q + 1)w_\pm^2$$
$$-2q(2p^2 + 3p + 1)w_\pm + q(p+1)^2][2p(w_\pm - 1)^2 + 2]^{-1} \tag{7.6}$$

In order to determine the direction with respect to the parameter w in which non-trivial relative equilibria branch off in the space (e_{y_3}, i_{33}, w) in the neighbourhoods of the bifurcation points $(-1, 1, w_\pm)$, it is necessary and sufficient to ascertain the sign of expression (7.6).

Consider the branching point $(-1, 1, w_-)$ (the branching point $(-1, 1, w_+)$). As was noted above, it exists for any physically possible values of the parameters p and q, i.e., for $(p, q) \in P$ (it exists for $(p, q) \in P_0$). Substituting the expression for $w_-(p, q)$ (for $w_+(p, q)$) given by formulae (7.5) into (5.5), we obtain a function that depends on two arguments: $z_- = z_-(p, q)$ $(z_+ = z_+(p, q))$. A computer analysis of this function revealed that it takes only positive values for $(p, q) \in P$ (for $(p, q) \in P_0$). Therefore, in the space (e_{y_3}, i_{33}, w) "oblique" relative equilibria branch off in the direction of increasing values of w in the neighbourhoods of the point $(-1, 1, w_-)$ (in the neighbourhoods of the point $(-1, 1, w_+)$), and, according to bifurcation theory, they are locally stable (locally unstable with a degree of instability equal to 2).

In order to determine the relative configuration of the rod and the axis of symmetry of the body on the branching-off permanent rotations, we ascertain the sign of the expressions for $x_\pm$, which are equal to ϑ/α in a first approximation (see formulae (7.1)). For this purpose, consider the first relation in (6.2):

$$i_{32}/e_{y_2} = q(w_\pm - 1)/w_\pm \tag{7.7}$$

Note that

$$i_{32} = -\sin\vartheta = -\vartheta + o(\vartheta), \quad e_{y_2} = \sin\alpha = \alpha + o(\alpha)$$

Consequently, in a first approximation $i_{32}/e_{y_2} = \vartheta/\alpha$. Hence,

$$x_\pm = q(1 - w_\pm)/w_\pm \tag{7.8}$$

Taking into account (see Section 5) that in the case of the trivial solution S^{-+} $w_- < 1 < w_+$ for $p > 0$ and $w_- < 1$ for $p < 0$, we conclude that $x_+ < 0$ (for $p > 0$) and that $x_- < 0$ (for any p). "Pendulum-like" permanent rotations branch off from the point $(-1, 1, w_-)$, and "regular-bump-like" permanent rotations branch off from the point $(-1, 1, w_+)$.

The oblique relative equilibria that branch off from the solutions S^{--}, S^{++} and S^{+-} are treated similarly. It turned out that in each of the regions P_0, P_1, P_2, P_3 and P_4 of the parameters p and q oblique relative equilibria branch off from trivial steady motions in the direction of larger values of w (i.e., $z_\pm > 0$ in all cases).

We will find the conditions under which oblique steady motions branch off in the space (i_{33}, e_{y_3}, k^2) at bifurcation points in a definite direction with respect to k^2, and we will thereby find the conditions for local stability of the permanent rotations.

Since the constants $k_\pm^2$ and $w_\pm$ for the trivial steady motions are related, according to formulae (5.1), by the relations $k_\pm^2 C^2 w_\pm g/l$, the analogous bifurcation points in the space i_{33}, e_{y_3}, k^2 will have the coordinates ±1, ±1, $k_\pm^2$. In the neighbourhoods of these bifurcation points, we have

$$k^2 = k_\pm^2 + y_\pm\alpha^2 + o(\alpha^2) \tag{7.9}$$

where $y_\pm$ denotes some constants that depend on the parameters of the system.

By analogy with the case of the oblique relative equilibria, the direction of the branching of the oblique steady motions with respect to k^2 is determined by the sign of $y_\pm$. The expression for $y_\pm$ can be obtained from Eq. (6.3) after substituting relations (7.1) and (7.9) into it.

On the basis of the theorem of the relation between the stability conditions of relative equilibria and steady motions (see Ref. 8), in some cases the direction in which non-trivial steady motions branch off can be determined *a priori*, and a conclusion can be drawn regarding the nature of the stability of these motions in the neighbourhoods of the bifurcation points.

Consider the points where the degree of instability of the solution S^{-+} changes. For $p < 0$ oblique steady motions are known to branch off from the solution S^{-+} at the single branching point $(-1, 1, k_-^2)$. According to what we have proved above, the relative equilibria branching off from the trivial steady motion S^{-+} at this point are locally stable. Therefore, the non-trivial steady motions branch off in the same direction as the relative equilibria and will also be locally stable.

For $p > 0$ oblique steady motions branch off from S^{-+} at two points $(-1, 1, k_-^2$ and $-1, 1, k_+^2)$ (see Fig. 5). According to what we have proved above, the relative equilibria branching off from the trivial steady motion S^{-+} at $(-1, 1, k_-^2)$ are locally stable, and, therefore, the non-trivial stationary motions will also be stable. At the point $(-1, 1, k_+^2)$ the oblique relative equilibria are unstable with a degree of instability equal to 2. However, steady motions cannot branch off in the other direction with respect to k^2, because in this case, according to bifurcation theory, they would be unstable with a degree of instability equal to 4, which is impossible.[8] Therefore, the non-trivial steady motions branch off in the same direction as the relative equilibria and will also be unstable with a degree of instability equal to 2.

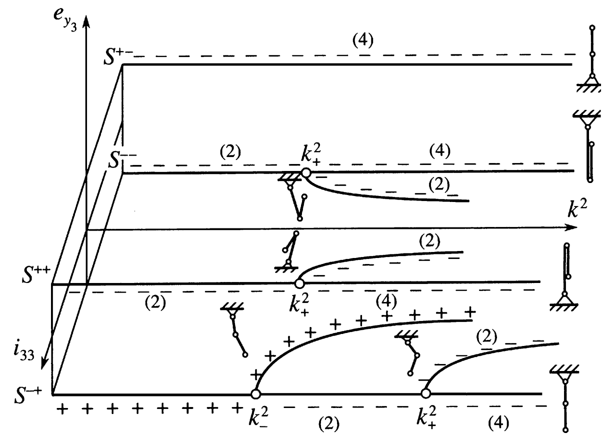

Fig. 5.

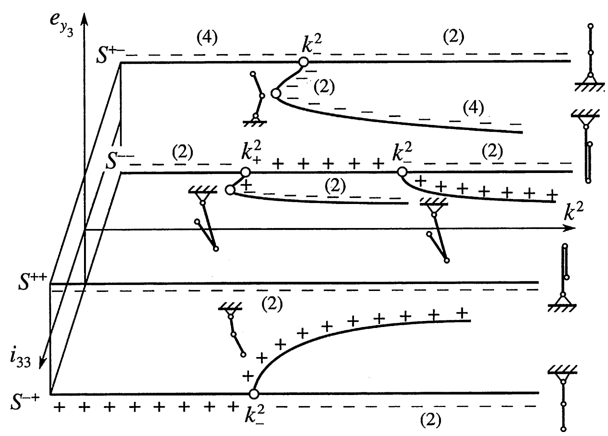

Fig. 6.

The steady motions that branch off from the trivial steady motions S^{--}, S^{++} and S^{+-} are examined in a similar manner. It turned out that, unlike the relative equilibria, for which $z_{\pm} > 0$ (see relations (7.1)), cases of steady motions for which $y_{\pm} > 0$ (see equalities (7.9)) are possible.

8. Bifurcation diagrams

The most complex types of Poincaré–Chetayev bifurcation diagrams of the steady motions of the mechanical system under consideration are plotted schematically in (e_{y_3}, i_{33}, k^2) space in Figs. 5–9. The initial portions of the oblique steady motions in the neighbourhoods of the branching points were continued to the extreme steady motions.

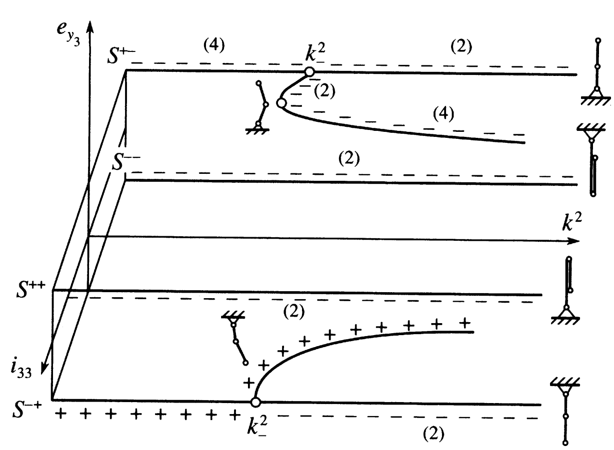

Fig. 7.

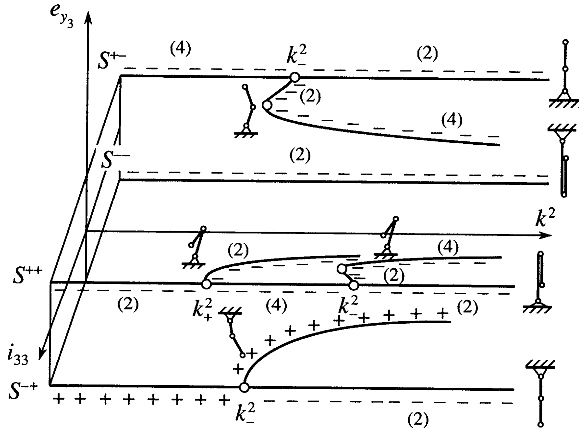

Fig. 8.

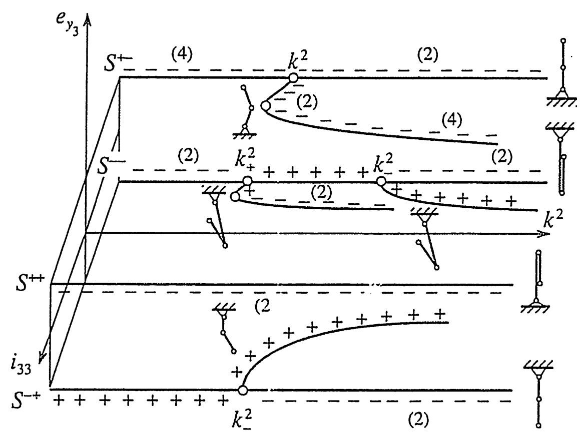

Fig. 9.

Fig. 5 shows the bifurcation diagram of steady motions in the case when $(p, q) \in P_0$. In this region there is a single bifurcation diagram. If $(p, q) \in P_1$, the system has three bifurcation diagrams of steady motions, one of which is shown in Fig. 6. On the other two bifurcation diagrams the steady motions emerging from the point $(-1, -1, k_+^2)$ of the family S^{++} branch off to the right, and the steady motions emerging from the point $(-1, -1, k_-^2)$ of the family S^{++} can branch off both to the right and to the left. Fig. 7 shows one of the two possible bifurcation diagrams of steady motions for the case of $(p, q) \in P_2$. On this diagram the physical parameters of the system are such that $y_- < 0$ for the point $(1, -1, k_-^2)$. The other diagram (not shown in Fig. 7), which corresponds to P_2, illustrates the case of $y_- > 0$ for the point $(1, -1, k_-^2)$. Fig. 8 shows one of the three possible bifurcation diagrams of steady motions in the case when $(p, q) \in P_3$. In this region oblique steady motions branch off from the solutions S^{-+}, S^{++} and S^{-+}. On the two bifurcation diagrams that are not shown the steady motions emerging from the point $(1, 1, k_-^2)$ of the family S^{++} branch off to the right, and the steady motions emerging from the point $(1, -1, k_-^2)$ of the family S^{++} can branch off both to the right and to the left. Finally, in the region P_4 four types of bifurcation diagrams are possible. One of them is shown in Fig. 8. In this region the values of the constant y_+ for the point $(-1, +1, k^2)$ are positive, and the values for the points $(-1, -1, k_+^2)$ and $(1, -1, k_-^2)$ can take both positive and negative values.

In all the figures from Fig. 5 to Fig. 9, the plus signs indicate stable permanent rotations, and the minus signs indicate unstable permanent rotations. The numbers in parentheses alongside the families of steady motions indicate the degree of instability of the respective permanent rotations. Conclusions regarding the stability of the steady motions were drawn on the basis of bifurcation theory.

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References

1. Moiseyev NN, Rumyantsev VV. *Dynamics of a Body with Cavities Containing a Fluid*. Moscow: Nauka; 1965.
2. Ishlinskii AYu, Storozhenko VA, Temchenko ME. *Rotation of a Rigid Body on a String and Related Problems*. Moscow: Nauka; 1991.
3. Rumyantsev VV. Dynamics of a rigid body suspended on a string. *Izv Akad Nauk SSSR MTT* 1983;(4):5–15.
4. Rubanovskii VN, Rumyantsev VV. Steady motions of a heavy symmetric rigid body suspended on a string. *Izv Akad Nauk SSSR MTT* 1985;(5):3–7.

5. Sarychev VA, Mirer SA. Steady motions of a body on a string suspension. In: *Non-Linear Mechanics*. Moscow: Fizmatlit; 2001. p. 281–322.
6. Rubanovskii VN. Branching and stability of permanent rotations and relative equilibria of a body suspended on a rod. *Prikl Mat Mekh* 1987;**51**(3):382–9.
7. Dosayev MZ, Samsonov VA. Stability of the rotation of a heavy body with a viscous filler. *Prikl Mat Mekh* 2002;**66**(3):427–33.
8. Karapetyan AV. *The Stability of Steady Motions*. Moscow: Editorial URSS; 1998.

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